A volume-preserving quasi-static free boundary problem

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Consider $u : \mathbb{R}^n \times [0,T) \to \mathbb{R}^+$. Suppose $\{u > 0\}$ is connected. Then u satisfies

$$(P) \qquad \begin{cases} -\Delta u(\cdot,t) = \lambda(t) & \text{in} \quad \{u > 0\}, \\ V = |Du|^2 - 1 & \text{on} \quad \partial\{u > 0\}, \end{cases}$$

where $\lambda(t) = \lambda(t; u)$ is the volume-preserving constant, chosen such that

$$\int u(\cdot,t)dx \equiv C.$$

The model is *motivated* by evolution of quasi-static droplets on the flat surface. In this context u denotes the height of the droplet, the positive phase $\{u > 0\}$ denotes the wet region and the free boundary $\partial\{u > 0\}$ denotes the contact line between the drop and the surface.

The second equation in (P) describes the contact line motion by a relationship between the free boundary normal velocity $V = u_t/|Du|$ and the "contact angle" $|Du|$.

Motivation

Associated energy of a droplet with surface height function $u(x)$:

$$E(u) = \int_{\{u>0\}} \gamma_{SL} - \gamma_{SV} + \gamma_{LV}(1 + \tfrac{1}{2}|Du|^2)dx$$

$$\text{(Laplace-Young): } \gamma_{SV} - \gamma_{SL} = \gamma_{LV}(1 - \theta_0^2)$$

$$= \int_{\{u>0\}} \gamma_{LV}(\theta_0^2 + \tfrac{1}{2}|Du|^2)dx$$

In the lubrication limit one obtains the thin-film equation

$$h_t = C\nabla \cdot [h^3\nabla(-\gamma_{LV}\Delta h)],$$

Our equation is obtained in the quasi-static limit: that is we assume that the contact line velocity is much slower than the capillary velocity. Many different formulas has been derived in the literature : Cox, Glasner, Tanner, Voinov...

We are interested in the global existence of a weak solution. Note that usual viscosity solution theory does not apply due to the nature of $\lambda(t; u)$.

Viscosity solutions

An idea of the definition:
An upper semi-continuous function $u(x,t) \geq 0$ is a *viscosity subsolution* of (FBP) if $u(x,t)$ cannot cross any classical solutions of (P) from below.

Similar definition for a *viscosity supersolution* (lower semi-continuous).

Definition $u(x,t)$ is a *viscosity solution* of a problem (FBP) if $u(x,t)$ is a viscosity supersolution and $u^*(x,t)$ is a viscosity subsolution with

$$u^*(x,t) = \limsup_{(y,s)\to(x,t)} u(y,s).$$

This is to embrace possible discontinuity of the solution.

• Viscosity solutions are well-posed for problems with *comparison principle* stated as below:

If v_1 and v_2 are two solutions of (FBP), and if $v_1(x,0) \leq v_2(x,0)$, then $v_1(x,t) \leq v_2(x,t)$.

For example, if $\lambda(t)$ in (P) were prescribed (not necessarily volume-preserving), then the comparison principle works and there exists a viscosity solution in global time, which is unique in a generic way.

I. Star-shaped initial data

For a single-component case, (P) can be approximated by freezing $\lambda(t)$ in each discrete-time interval of size h and solving for corresponding viscosity solution. This way one can find an approximate solution u_h.

To gain control the change of $\lambda(t)$ over time, we solve the problem $(P)^M$ with
$V = \max(|Du|^2 - 1, M)$, to get an approximate solution u_h^M.

We impose the following geometric condition (I):

$(I1)$ $\{u(\cdot, 0) > 0\}$ is star-shaped with respect to a small ball $B_r(0)$.

$(I2)$ $B_r(0) \subset \{u_h^M(\cdot, t) > 0\}$ for $0 \leq t \leq T$

Under the hypothesis (I), for $0 < k \leq h$ we have

$$A(t)^{-n} u_h^M(A(t)x, t) \leq u_k^M(x, t) \leq A(t)^n u_h^M(x/A(t), t)$$

where $A(t) := 1 + Ae^{At}h$.

This gives us an estimate between the discrete and continuous solutions.

Theorem 0.1 (Glasner-K). *Under hypothesis (I), the following holds:*

(a) $\{u_h^M(\cdot,t) > 0\}$ *is star-shaped with respect to* $B_r(0)$ *for* $0 \le t \le T$.

(b) u_h^M *converges to* u^M*, a unique solution of the problem* $(P)^M$.

(c) (along a subsequence,) u_h^M *and corresponding* $\lambda_h^M(t)$ *locally uniformly converges to* u *and* $\lambda(t)$ *in space and time.*

Then u *is a viscosity solution of* (P) *with* $\lambda(t)$*,* $\{u(\cdot,t) > 0\}$ *is star-shaped, and* $\int u(\cdot,t)dx \equiv M$ *for* $0 \le t \le T$.

Q: When is the hypothesis (I) true?

As a corollary, the following holds:

1. Suppose the initial data $u_0(x)$ is star-shaped with respect to a small ball $B_r(x_0)$. Then there exists a "viscosity" solution $u(x,t)$ of (P) for some time $[0, t_0]$ with $u(x,0) = u_0(x)$.

2. Solutions exist at all times if

(a) Ω_0 is star-shaped and is symmetric with respect to two orthogonal axes

or

(b) $n = 2$ and Ω_0 is convex and symmetric to one axis.

Note that the problem does not preserve star-shaped positive phase, at least not with fixed center.

Q: Is the problem convexity-preserving?

Q: Does the positive phase get round after a long time?

Even with star-shaped initial data, we do not know whether $\{u > 0\}$ will stay as a single component except in a few specific settings.

• In general, due to topological changes of the positive phase, $\{u > 0\}$ consists of multiple components. In this case we assign different $\lambda(t)$ in each connected component of the positive phase to preserve the volume of the droplet in each component.

• In the event of merging, we pick $\lambda(t)$ for the merged components such that the volume from the merging is preserved. (Note that in this case the value of λ may jump at the merging time.

II. General case: variational approach [Grunewald-K]

Here we introduce a discrete approximation scheme, based on the gradient flow structure of (P).

The goal is to construct approximate solutions generated with discrete-time minimization process, which converges to a weak solution of (P).

The motivation comes from Almgren-Taylor-Wang's approach on mean curvature flow, and also later work by Alt-Caffarelli, Chambolle, Cardiliaguet-Ley, Luckhaus, Röger, ...

Let us define the energy

$$E(\omega) := \int_{\omega} |\nabla u_{\omega}|^2 \, dx \ + |\omega|$$

where u_{ω} minimizes the first term in the energy with given ω with the Dirichlet boundary condition and the volume-preserving condition.

Let us also define the"distance"

$$\begin{aligned} \tilde{d}(\omega_0, \omega_1) \ &:= \ \int_{\omega_0 \Delta \omega_1} dist(x, \partial\omega_0) \, dx \\ &\sim \ \int |\chi_{\omega_1} - \chi_{\omega_0}|^2 dx \end{aligned}$$

(Note that this is NOT a distance.)

Given ω_{kh}, we minimize

$$\frac{1}{h}\widetilde{dist}^2(\omega, \omega_0) + E(\omega) + \epsilon|\partial\omega|$$

among the sets of finite perimeters (necessitating the third term) to get the next set $\omega_{(k+1)h}$.

We then define

$$u_h(\cdot, t) := u_{\omega_{kh}} \quad \text{for } kh \le t < (k+1)h.$$

Formally, in the limit $h \longrightarrow 0$, one obtains the solution to

$$\begin{cases} -\Delta u = \lambda(u) & \text{in } \{u > 0\} \\ V = |Du|^2 - 1 + \epsilon\kappa & \text{on } \partial\{u > 0\} \end{cases}$$

If we set $\epsilon = h$ then we recover the original problem.

Remarks on regularity properties of the discrete set evolution:

○ The set evolution $\{\omega_{kh}\}, k = 1, \ldots$ is equicontinuous over time (with respect to $\tilde{d}$) when $\epsilon > 0$, following the argument in [Luckhaus-Sturzenhecker] and [Luckhaus] in the case of the mean curvature flow and Stefan problem with surface tension. This equicontinuity is uniform in h but local in time.

○ Obtaining a spatial regularity on ω_{kh} uniform in h is a challenging task in general.

○ For a fixed time step h, and for a similar problem corresponding to fixed λ, a uniform Lipschitz regularity of ω_{kh} is shown in [Alt-Caffarelli, 1981].

Barrier property for the time–discrete solution. (Subsolution property)

Suppose there exists a smooth function ϕ with $|D\phi| \neq 0$ in $B_r(x_0) \times [0, h]$ such that

$$\begin{cases} -\Delta\phi & < \min(\lambda_0, \lambda_h) \\ \frac{\phi_t}{|D\phi|} - (|D\phi|^2 - 1 - h\kappa_\phi) & < 0 \text{ in } B_r(x_0) \times [0, h]. \end{cases}$$

Then for sufficiently small h, the following holds:

If $\phi \prec u_h$ in $B_r(x_0) \times \{t = 0\}$ and on $\partial B_r(x_0) \times \{t = h\}$, then $\phi \prec u_h$ in $B_r(x_0) \times \{t = h\}$.

Corresponding supersolution property is also satisfied.

In other words, the solution u_h behaves similar to the discrete viscosity solution with time step h.

Convergence as $h \to 0$

If $\lambda(t; u)$ were prescribed, then the discrete solutions converge uniformly to a corresponding viscosity solution.

For our problem, λ depends on time and the components, and therefore we should consider λ as a function of (x, t).

If $\lambda_h(x, t)$ locally uniformly converges, then so does u_h.

General convergence result

(a) The set $w_*(x,t) := \liminf_{(y,s)\to(x,t)}\{u_h(y,s) > 0\}$ is a supersolution of (P) with prescribed multiplier $\lambda_*(x,t) := \liminf_{(h,y,s)\to(0,x,t)} \lambda_h(y,s)$;

(b) The set $w^*(x,t) := \limsup_{(h,y,s)\to(0,x,t)}\{u_h(y,s) > 0\}$ is a subsolution of (P) with prescribed multiplier $\lambda^*(x,t) := \limsup_{(h,y,s)\to(0,x,t)} \lambda_h(y,s)$.

In particular if $\lambda(x,t)$ uniformly converges it follows that u_h uniformly converges to the unique viscosity solution with prescribed λ.

Literature: Mean curvature flow

Almgren-Taylor-Wang (1993) and Luckhaus-Sturzenhecker(1995) studied the mean curvature flow with parallel approach. In this case

$$E(\omega) = |\partial\omega|.$$

In [ATW] a version of barrier property is proven for discrete solutions based on regularity theory of minimal surfaces. Using this, Chambolle (2004) has shown that the limit solution of the discrete time solutions coincides with the viscosity solution of the mean curvature flow.

Literature: Other interface problems

• Cardaliaguet and Ley (2006): Convergence result of discrete solutions for the free boundary problem

$$\begin{cases} u = 1 & \text{on} \quad S \times [0, \infty) \\ -\Delta u = 0 & \text{in} \quad \{u > 0\} \\ V = |Du|^2 - 1 & \text{on} \quad \partial\{u > 0\} \end{cases}$$

is proved, based on the regularity results of [Alt- Caffarelli, 1981] for minimizers of the type $\int (|Du|^2 + f)\chi_{\{u>0\}}$. This problem satisfies the comparison principle.

• Luckhaus (1991), Röger (2004): global time existence result for two-phase Stefan problem with surface tension. Here the surface tension effect generates sufficient regularity for the time-discrete solutions to yield convegence result to a weak and non-unique (varifold) solution.

Back to the single component case: variational and viscosity solutions

Let us now put an additional constraint on the minimization process such that the new set $\omega_{(k+1)h}$ to be contained in the Mh-neighborhood of the old set ω_{kh}.

Then the corresponding solution u_h^M satisfies barrier properties corresponding to $(P)^M$. This enables the comparison with viscosity solutions:

Theorem 0.2. *Let u^M be the unique viscosity solution of $(P)^M$ for $0 \leq t \leq T$ with star-shaped support with respect to $B_r(0)$. Then u_h^M uniformly converges to u^M. Moreover for some A depending on M,*

$$A(t)^{-n} u^M(A(t)x, t) \leq u_h^M(x,t) \leq A(t)^n u^M(x/A(t), t)$$

where $A(t) = (1 + Ae^{At}h)$.

It follows that the energy

$$E(\Omega_t) = |\Omega_t| + \int |Du_{\Omega_t}|^2$$

decreases for $\Omega_t := \{u^M(\cdot, t) > 0\}$. The same holds for its (subsequential) local uniform limit u: a viscosity solution of (P).

Similarly, we can compare a smooth solution with u_h^M to show:

Theorem 0.3. *If a classical solution $\Phi(x,t)$ of (P) exists for $0 \le t \le T$ such that Φ can be extended to a smooth function $\Psi(x,t) \in [-\epsilon, \infty)$, then there exists an $A > 0$ such that, for sufficiently small $h > 0$,*

$$A(t)^{-n}(\Phi(x,t) - A(t)h)_+ \le u_h^M \le A^n(t)(\Psi(x,t) + A(t)h)_+$$

where $A(t) = (1 + Ae^{At}h)$. In particular u_h^M converges locally uniformly to Φ as $h \to 0$, if M is sufficiently large.

This indicates that the discrete scheme is "reasonable".