# Evolution of a relativistic string

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# Minimal surfaces in Minkowski space

The evolution of a classical relativistic string is usually modelled by the (hyperbolic) equation

$$\mathrm{a} = (1 - \mathrm{v}^2)\kappa \qquad (1)$$

which is the geometric analog of the well–known Born-Infeld equation in the case of graphs.

Equation (1) also corresponds to the minimal surface equation for timelike submanifolds of the Minkowski space $\mathbb{R}^{1+n}$.

Equation (1) has been considered by many authors, and short-time existence has been established by T. Deck and O. Müller when $n = 2$ (strings), and by O. Milbredt in the general case.

The issue of global existence is more delicate, since the equation develops singularities in finite time. However, when the initia data is almost flat, global existence has been proved by H. Lindblad and later extended by S. Brendle.

# The case of strings

When $n = 2$ a minimal surface, which we also call a relativistic string, can be parametrized by a function $\gamma : [0, T] \times [0, L] \to \mathbb{R}^{1+2}$ solving the linear wave equation

$$\gamma_{tt} = \gamma_{xx},$$

with constraints

$$\gamma_t \cdot \gamma_x = 0 \qquad |\gamma_t|^2 + |\gamma_x|^2 = 1.$$

In particular, one has the explicit solution

$$\gamma(t, x) = \frac{a(x+t) + b(x-t)}{2},$$

where $a, b : [0, L] \to \mathbb{R}^2$ satisfy $|a'| = |b'| = 1$.

Such representation implies that minimal surfaces are not closed under Hausdorff convergence.
Indeed, there are sequences $a_\epsilon \to a$ and $b_\epsilon \to b$ such that $|a'| \leq 1$, $|b'| \leq 1$, but the equality does not hold, so that

$$\gamma_\epsilon(t,x) = \frac{a_\epsilon(x+t) + b_\epsilon(x-t)}{2} \to \gamma(t,x) = \frac{a(x+t) + b(x-t)}{2}$$

where $\gamma$ satisfies the wave equation but not the minimal surface equation.

Such phenomenon is known in the physical literature under the name of wiggly strings (see for instance Vilenkin & Shellard, 1994).

# A closure result

We call subrelativistic string a surface which can be parametrized by a function $\gamma$ solving the linear wave equation and such that $|a'| \leq 1$, $|b'| \leq 1$.

Theorem [Bellettini, Hoppe, N., Orlandi]

The subrelativistic strings are the closure of the relativistic strings under Hausdorff convergence.

An analogous result was previuosly obtained by Y. Brenier in the case of graphs, using a different parametrization.

The result was already stated in the book by Vilenkin and Shellard.

# Singularities

The formation of singularities is discussed in Vilenkin & Shellard, and further analyzed in a recent paper by J. Eggers and J. Hoppe, where it is shown that a singularity is generically a cusp, with the asymptotic profile $y \sim x^{\frac{2}{3}}$ in graph coordinates.

It is controversial if the previous representation can be used to define reasonable weak solutions to the minimal surface equation, after the onset of singularities.

# Convex strings

However, the formation of a cusp is not a generic singularity for centrally symmetric convex strings.

Theorem [Bellettini, Hoppe, N., Orlandi]
Assume that the initial string is a centrally symmetric convex curve, with zero velocity. Then the evolution reamins convex and encounters an extinction singularity. Moreover, if the initial curve is smooth, the limit shape is a round circle.

Remark

If the initial convex curve is not smooth, for ex. a square, then the limit shape is not necessarily a circle.

It is unclear is such result still holds in higher dimensions.

A similar result has been established by D. Kong, L. Kefeng and Z. Wang for the hyperbolic curvature flow of planar curves.

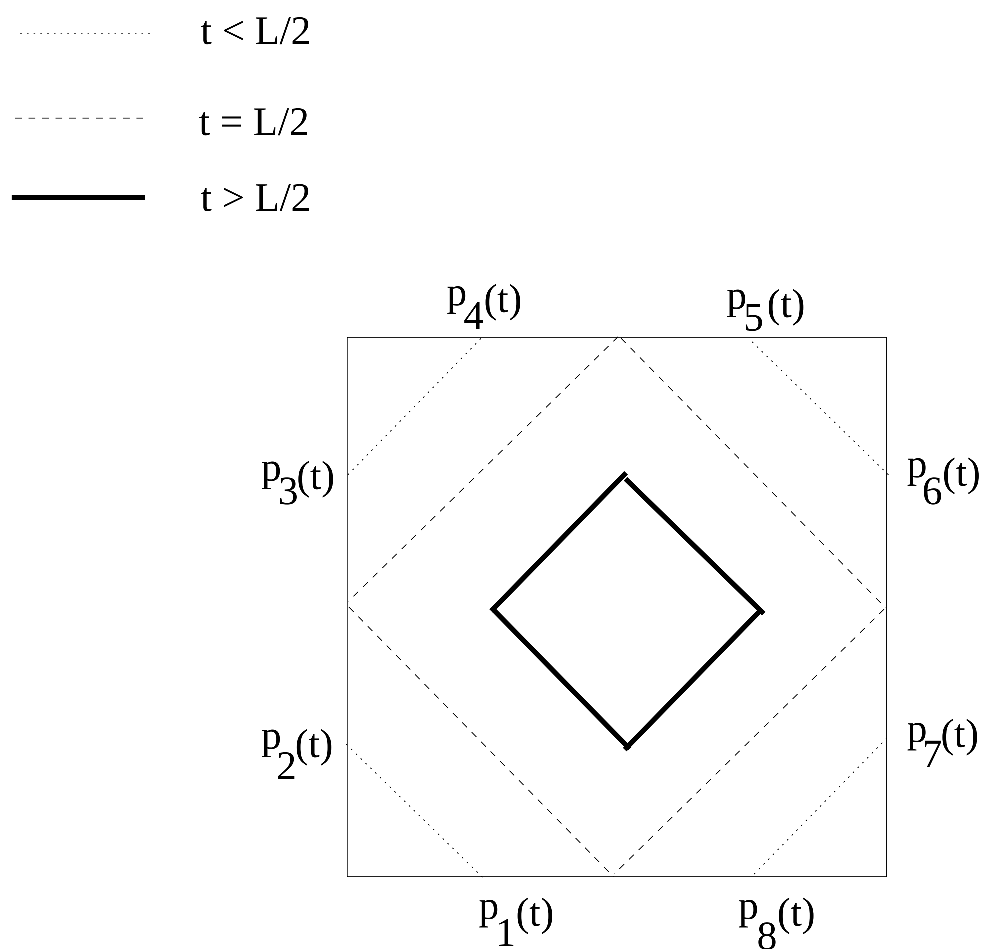


Figure 1: an evolving square.

# Generalized minimal surfaces

A Radon measure $V$ on $\mathbb{R}^{1+n} \times G_{n+1-k}$ is a rectifiable lorentzian varifold of codimension $k$ if $\Gamma = \operatorname{spt}(\pi_{\#} V)$ is a $(n+1-k)$-dimensional rectifiable set whose tangent space is timelike almost everywhere, and

$$\int_{\mathbb{R}^{1+n} \times G_{n+1-k}} \varphi(x, P)\, dV(x, P) = \int_{\Gamma} \varphi(x, T_x\Gamma)\theta(x)\, d\mathcal{H}^{n+1-k}(x)$$

for some $\theta > 0$ and for all $\varphi \in C_c(\mathbb{R}^{1+n} \times G_{n+1-k})$.

The varifold $V$ is stationary if

$$\int_\Gamma \mathrm{tr}\,(P\nabla X)\,\theta\,d\mathcal{H}^{n+1-k}$$

where $P$ is the lorentzian orthogonal projection onto $T_x\Gamma$ and $X \in \left(C_c^1(\mathbb{R}^{1+n})\right)^{1+n}$.

When $\Gamma = \mathrm{spt}(\pi_\# V)$ is smooth and $T_x\Gamma$ is timelike, a direct computation shows that the stationarity condition implies that $\Gamma$ is a minimal submanifold of codimension $k$ and $\pi_\# V$ coincides, up to a positive constant, with the $(n+1-k)$-dimensional Minkowski area restricted to $\Gamma$.

# Approximation with nonlinear waves

As in the elliptic and parabolic setting, one may try to approximate minimal surfaces by means of solutions of the semilinear wave equation

$$u_{tt} - \Delta u + \frac{1}{\epsilon^2} W'(u) = 0 \tag{2}$$

where $u : \mathbb{R}^{1+n} \to \mathbb{R}^k$, with $k \in \{1, 2\}$, and $W$ is a double–well potential.

A formal asymptotic analysis which confirms such expectation has been performed by J. Neu.

The singular perturbation limit has been studied by R. Jerrard and F.H. Lin when $n = k = 2$ (vortices).

In a recent work, R. Jerrard establishes convergence for small time of (the zero-level set of) solutions of (2) to minimal surfaces, in the case of well–prepared initial data.

Understanding convergence after the singularities and for more genral initial data seems to be a difficult task.

It is possible to associate to a solution $u_\epsilon$ of (2) a stationary varifold $V_\epsilon$ which is expected to concentrate in the limit on a minimal surface (and it does for well–prepared initial data, as proved by R. Jerrard).

Theorem [Bellettini, N., Orlandi]

Under suitable assumption on the density and the tangent space of the limit varifold $V = \lim_\epsilon V_\epsilon$, implying in particular rectifiability, we have that $V$ is a stationary rectifiable lorentzian varifold.

This result is analog to a result by L. Ambrosio and H.M. Soner in the parabolic setting, even if it holds under stronger assumptions.

# Stress-energy tensor

$$
\begin{aligned}
e_\epsilon &= c_k(\epsilon)\left(\frac{|u_{\epsilon t}|^2+|\nabla u_\epsilon|^2}{2}+\frac{W(u_\epsilon)}{\epsilon^2}\right)\\
\ell_\epsilon &= c_k(\epsilon)\left(\frac{-|u_{\epsilon t}|^2+|\nabla u_\epsilon|^2}{2}+\frac{W(u_\epsilon)}{\epsilon^2}\right)\\
T_\epsilon^{\alpha\beta} &= -c_k(\epsilon)\eta^{\alpha\gamma}\partial_{x^\gamma}u_\epsilon\cdot\eta^{\beta\delta}\partial_{x^\delta}u_\epsilon+\ell_\epsilon\,\eta^{\alpha\beta}
\end{aligned}
$$

where $c_1(\epsilon)=\epsilon$, $c_2(\epsilon)=|\log\epsilon|^{-1}$, and $\eta^{\alpha\beta}$ is the standard metric tensor in Minkowski space. As $\epsilon\to 0$, we have

$$
e_\epsilon\,dt\,dx \rightharpoonup e \qquad \ell_\epsilon\,dt\,dx \rightharpoonup \ell
$$

$$
c_k(\epsilon)\,\frac{W(u_\epsilon)}{\epsilon^2} \rightharpoonup w \qquad T_\epsilon\,dt\,dx \rightharpoonup T.
$$

# Assumptions on initial data

We assume that there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^n} e_\epsilon(0,x)\,dx \;\leq\; C$$

for all $\epsilon$.

Notice that

$$\int_{\mathbb{R}^n} e_\epsilon(t,x)\,dx \;=\; \int_{\mathbb{R}^n} e_\epsilon(0,x)\,dx \qquad \text{conservation of energy}$$

for all $t > 0$

# Assumptions on the limit varifold

Let

$$\widetilde{T}^{\alpha\beta} = \frac{d\,T^{\alpha\beta}}{d\,\ell}.$$

(A1) For $\ell$-a.e. $(t,x)$ it holds

$$0 < \lim_{\rho\to 0} \frac{\ell(B_\rho(t,x))}{\rho^{n+1-k}} < +\infty\,.$$

(A2) For $\mathcal{H}^{n+1-k}$-a.e. $x \in \Gamma = \mathrm{spt}(\pi_{\#}V)$, the tensor $\mathrm{Id} - \eta\widetilde{T}(x)$ is spacelike, that is $\left(\mathrm{Id} - \eta\widetilde{T}(x)\right)\xi$ is spacelike for all $\xi$.

(A3) When $k = 1$

$$\frac{d\,w}{d\,\ell} = \frac{1}{2}\,.$$

Assumption (A1) ensures that the lagrangian integrands $\ell_\epsilon$ concentrate on a rectifiable set of codimension $k$ (see Preiss and Ambrosio–Soner).

Assumption (A2) is equivalent to require that the normal vector to $\Gamma$ is spacelike almost everywhere.

Assumption (A3) corresponds to the so-called equipartition of energy, which holds true in the elliptic and parabolic setting (see T. Ilmanen), but could fail in the hyperbolic case.

However, differently from the elliptic and the parabolic case, this result cannot be true without any assumption, since in general the limit varifold $V$ is not rectifiable (while it is necessarily stationary), as shown by the previous examples when $n = 2$.

It would be interesting to understand what is the natural generalization of subrelativistic strings in higher dimension, and how the minimal surface equation is relaxed under Hausdorff or varifold convergence.