Representation Theory and Probability

Part I - Representation theory, Littelmann's path model and probability

Pioneers : Ph Biane, Ph. Bougerol, N. O'Connell

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$\mathcal{P}(B) \stackrel{d}{=}$ Brownian motion conditioned (in Doob's sense) to remain positive

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I-Conditioned Markov Processes

- $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space
  $(X_t)_{t\geq 0}$ a Markov process with values in E,
  transition semi-group $(P_t)_{t\geq 0}$
  $(\theta_s)_{s\geq 0}$ the time shift operators, $X_t \circ \theta_s = X_{t+s}$, $t \geq 0$.
  $\mathcal{I} = \{A \in \mathcal{F} : \forall s \geq 0,\ \theta_s^{-1}(A) = A\}$

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- Let $A \in \mathcal{I}$
  Consider the harmonic function $h(x) = \mathbb{P}_x(A)$, $x \in E$ $(\mathbb{E}_x(h(X_t)) = h(x))$.
  Suppose $\exists x_0 \in E$, $h(x_0) \neq 0$.
  Define a probability $\hat{\mathbb{P}}_{x_0}$ on Ω

$$\hat{\mathbb{P}}_{x_0}(.) = \frac{1}{h(x_0)} \mathbb{P}_{x_0}(. \cap A).$$

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  Notice that

$$\hat{\mathbb{P}}_{x_0}(B) = \mathbb{E}_x\Big(\frac{h(X_t)}{h(x_0)}1_{B\cap A}\Big), \quad B \in \mathcal{F}_t,$$

  where $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration of $(X_t)_{t\geq 0}$.

I-Conditioned Markov Processes

- Under $\hat{\mathbb{P}}_{x_0}$, $(X_t)_{t\geq 0}$ is a Markov process with transition probability semi-group

$$\hat{P}_t(x, dy) = \frac{h(y)}{h(x)} P_t(x, dy), \quad x, y \in \hat{E},$$

where $\hat{E} = \{x \in E : h(x) \neq 0\}$.

- Doob's transform of P_t, h-transform ...

Classical examples :

- Let $q > 1$ and $(X_n)_{n\geq 0}$ be a simple random walk on $\mathbb{Z}$ with positive drift, with Markov kernel

$$K(x,y) = \frac{q^{y-x}}{q+q^{-1}} 1_{|y-x|=1}, \quad x,y \in \mathbb{Z}.$$

Let $T = \inf\{n : X_n \leq -1\}$.

The event $\{T = +\infty\}$ is shift invariant for the killed process $(X_{n \wedge T})$.

The function $h(x) = \mathbb{P}_x(T = +\infty)$ is harmonic for the killed process $(X_{n \wedge T})$ and vanishes on $\mathbb{Z}_-^*$.

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One has

$$h(x) = Cq^{-x}s_x(q)1_{x\geq -1},$$

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The process conditioned not to be killed has a Markov kernel

$$\hat{K}(x,y) = \frac{s_y(q)q^{-y}}{s_x(q)q^{-x}} K(x,y) = \frac{s_y(q)}{s_x(q)s_1(q)} 1_{|y-x|=1}, \quad x, y \in \mathbb{N}.$$

When $q \to 1$, the conditioned process converges toward the process with Markov kernel

$$\hat{K}(x,y) = \frac{y+1}{2(x+1)} 1_{|y-x|=1}, \quad x, y \in \mathbb{N},$$

i.e., the simple symmetric random walk conditioned, in the Doob's sense, to remain non negative.

I-Conditioned Markov Processes

- Let $(B_t)_{t\geq 0}$ be a Brownian motion on $\mathbb{R}$ with drift $\gamma > 0$, with semi-group $(p_t)_{t\geq 0}$. The Brownian motion conditioned to remain positive has semi-group

$$p_t(x, y) = \frac{1 - e^{-2\gamma y}}{1 - e^{-2\gamma x}} p_t^0(x, y),\ x, y \in \mathbb{R}_+$$

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When $\gamma \to 0$, one gets the standard Brownian motion conditioned, in the Doob's sense, to remain positive.

1) Representation of $\mathfrak{sl}(2,\mathbb{C})$

- $\mathfrak{sl}(2,\mathbb{C}) = \{x \in M_2(\mathbb{C}) : \mathrm{tr}(x) = 0\} = \mathbb{C}e \oplus \mathbb{C}f \oplus \mathbb{C}h$.
  $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

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- Complex Lie algebra : Vector space over $\mathbb{C}$ equipped with a Lie bracket $[x,y] = xy - yx$, $x, y \in \mathfrak{sl}(2,\mathbb{C})$.

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- Complex Lie algebra : Vector space over $\mathbb{C}$ equipped with a Lie bracket $[x, y] = xy - yx$, $x, y \in \mathfrak{sl}(2,\mathbb{C})$.
- Commutation relations : $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$.

- A finite dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$ is a pair (ρ, V) where
  1. V is a finite dimensional vector space over $\mathbb{C}$,
  2. $\rho : \mathfrak{sl}(2,\mathbb{C}) \rightarrow \mathrm{End}(V)$ is a homomorphism of Lie algebras :
     a linear map and $[\rho(x), \rho(y)] = \rho[x, y]$, $x, y \in \mathfrak{sl}(2,\mathbb{C})$.

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- (ρ, V) is irreducible if for every linear subspace $W \subset V$,
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- Two representations (ρ_1, V_1) and (ρ_2, V_2) are isomorphic if
  $\exists$ an isomorphism $\varphi : V_1 \to V_2$ such that
  $\forall x \in \mathfrak{sl}(2,\mathbb{C}),\ \rho_2(x) \circ \varphi = \varphi \circ \rho_1(x)$.

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where each V_i is irreducible. The decomposition is unique (up to isomorphisms of representations).

- An irreducible finite dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$ is isomorphic to a representation V_λ, $\lambda \in \mathbb{N}$.
- $V_\lambda = \text{span}\{v_0, \dots, v_\lambda\}$. For $i \in \{1, \dots, \lambda\}$,

$$\rho(h)v_i = (\lambda - 2i)v_i, \ \rho(f)v_i = (i+1)v_{i+1}, \ \rho(e)v_i = (\lambda - i + 1)v_{i-1},$$

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  with $v_{-1} = v_{\lambda+1} = 0$.
- Terminology :
  1. $\lambda - 2i$: weight of v_i.
  2. λ : highest weight.
  3. V_λ : irreducible representation with highest weight λ.

- Character of a representation (ρ, V) : $\mathrm{ch}_V(q) = \mathrm{tr}(q^{\rho(h)})$, $q \in \mathbb{R}_*$.

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- (ρ_1, V_1) and (ρ_2, V_2) are isomorphic $\Leftrightarrow$ $\mathrm{ch}_{V_1} = \mathrm{ch}_{V_2}$.
- (Weyl character formula)

$$\begin{aligned} \mathrm{ch}_{V_\lambda}(q) &= q^{\lambda} + q^{\lambda-2} + \cdots + q^{-\lambda} \\ &= \frac{q^{\lambda+1} - q^{-(\lambda+1)}}{q - q^{-1}} \\ &= s_\lambda(q). \end{aligned}$$

- For two representations (ρ_1, V_1), (ρ_2, V_2) one defines a representation $(\rho, V_1 \otimes V_2)$ letting for $x \in \mathfrak{sl}(2,\mathbb{C})$

$$\rho(x)(v_1 \otimes v_2) = \rho(x)(v_1) \otimes v_2 + v_1 \otimes \rho_2(x)(v_2), \quad v_1 \in V_1, v_2 \in V_2.$$

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equivalent to

$$s_\lambda s_\beta = \sum_\mu c^\mu_{\lambda,\beta} s_\mu.$$

Examples

- $$\begin{aligned} V_1 \otimes V_1 &= \text{span}\{v_0 \otimes v_0, v_0 \otimes v_1 + v_1 \otimes v_0, v_1 \otimes v_1\} \oplus \mathbb{C}(v_0 \otimes v_1 - v_1 \otimes v_0), \\ &\simeq V_2 \oplus V_0 \end{aligned}$$

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  The isomorphism is equivalent to $s_1 s_1 = s_2 + s_0$

- Glebsch-Gordan rule,

$$\lambda \in \mathbb{N}, \quad s_\lambda s_1 = s_{\lambda+1} + s_{\lambda-1}, \qquad (s_{-1} = 0)$$

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- Towards a path model ...

2) Littelmann paths model

- A path π (of size n) is an application

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- Weight of a path π of size n

$$w(\pi) := \pi(n).$$

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- Properties :
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  3. if $f\pi \neq 0$ then $ef\pi = \pi$
  4. if π is a dominant path and $\pi(n) = \lambda$ then the smallest n such that $f^n\pi = 0$ is $\lambda + 1$.

Write π_λ for a dominant path ending at $\lambda \in \mathbb{N}$.

- Littelmann module generated by a dominant path π_λ

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- $q \in \mathbb{R}_*$,

$$\begin{aligned} \mathrm{ch}_{B\pi_\lambda}(q) &:= \sum_{\pi \in B\pi_\lambda} q^{w(\pi)} \\ &= \mathrm{ch}_{V_\lambda}(q) = s_\lambda(q) \end{aligned}$$

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- $\lambda, \beta \in \mathbb{N}$

$$B\pi_\lambda * B\pi_\beta = \sqcup B\pi_\mu,$$

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- equivalent to

$$s_\lambda s_\beta = \sum s_\mu$$

- thus $c^\mu_{\lambda,\beta}$=the number of dominant paths ending at μ.

A path π is in the Littelmann module generated by π_λ

$$\Leftrightarrow$$

For all $t \in [0, n]$,

$$\begin{aligned}\mathcal{P}(\pi)(t) &:= \pi(t) - 2 \inf_{0 \leq i \leq t} \{\pi(i)\} \\ &= \pi_\lambda(t).\end{aligned}$$

- $\pi_1(t) = t, \ t \in [0, 1]$.

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- Let $q \in \mathbb{R}+^*$, μ be a probability measure on $B\pi_1$.

$$\mu(\pi_1) = \frac{q^{w(\pi_1)}}{q+q^{-1}}, \quad \mu(f\pi_1) = \frac{q^{w(f\pi_1)}}{q+q^{-1}}.$$

III-Littelmann's path model and Pitman's theorem

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- $X(t) = x_0(1) + \cdots + x_{n-1}(1) + x_n(t-n),\ t \in [n, n+1]$.
- $\mathbb{P}(X(t) = x(t), t \in [0,k]) = \frac{q^{x(k)}}{s_1(q)^k}$.

Theorem : $(\mathcal{P}Y(n), n \geq 0)$ is a Markov chain with transition kernel

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Corollary : $(\mathcal{P}Y(n), n \geq 0)$ is distribued as a simple random walk with drift $\frac{q-q^{-1}}{q+q^{-1}}$ conditioned to remain non negative.

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- $(\mathcal{P}(B)(t), t \geq 0)$ as the same law as a standard Brownian motion conditioned to remain postive.

Limit object when $q = e^{\frac{\gamma}{\sqrt{n}}}$, $\gamma > 0$.

- $\frac{1}{\sqrt{n}}\mathcal{P}(X)(nt) \to \mathcal{P}(B^{\gamma})(t)$

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- $\frac{1}{\sqrt{n}}\mathcal{P}(X)(nt) \to \mathcal{P}(B^\gamma)(t)$
- $(\mathcal{P}(B^\gamma)(t), t \geq 0)$ as the same law as a standard Brownian with drift γ motion conditioned to remain postive.